

A FEW CLASSES OF INFINITE SERIES IDENTITIES FROM A MODULAR TRANSFORMATION FORMULA

SUNG GEUN LIM

ABSTRACT. The author proved a modular transformation formula for a function related to generalized non-holomorphic Eisenstein series and, using this formula, established a lot of infinite series identities. In this paper, we find more generalized series relations which contain the author's previous work.

1. Introduction

B. C. Berndt established a lot of infinite series identities from a modular transformation formula for a function coming from generalized Eisenstein series([2, 3]). Especially some of identities which he found are generalizations of Ramanujan's formulae in [7]. Following his work, the author considered a more general function which stems from generalized non-holomorphic Eisenstein series([4]) and, using a modular transformation formula for this function, established a lot of infinite series identities([5, 6]). In this paper, we find more generalized infinite series identities.

We shall introduce the following notations. Let \mathbb{C} be the set of complex numbers. The branch of the argument for $z \in \mathbb{C}$ is defined by $-\pi \leq \arg z < \pi$. For any non-negative integer n , the rising factorial $(x)_n$ is defined by

$$(x)_n = x(x+1) \cdots (x+n-1) \text{ for } n > 0, (x)_0 = 1.$$

Let $\Gamma(s)$ be the Gamma function. The confluent hypergeometric function of the first kind ${}_1F_1(\alpha; \beta; z)$ is defined by

$${}_1F_1(\alpha; \beta; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n n!} z^n$$

Received June 13, 2022; Accepted October 25, 2022.

2010 Mathematics Subject Classification: Primary 34A25; Secondary 11M36.

Key words and phrases: Infinite series identities, Modular transformation.

and the confluent hypergeometric function of the second kind $U(\alpha, \beta, z)$ is defined to be

$$U(\alpha; \beta; z) = \frac{\Gamma(1 - \beta)}{\Gamma(1 + \alpha - \beta)} {}_1F_1(\alpha; \beta; z) + \frac{\Gamma(\beta - 1)}{\Gamma(\alpha)} z^{1-\beta} {}_1F_1(1 + \alpha - \beta; 2 - \beta; z).$$

Let ${}_2F_1(\alpha, \beta; \gamma; z)$ be a hypergeometric function defined by

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} z^n,$$

which has the following integral representation, for $\text{Re}(\gamma) > \text{Re}(\alpha) > 0$ and $z \in \mathbb{C} \setminus [1, \infty)$ ([1], p. 65),

$${}_2F_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 t^{\alpha-1} (1-t)^{\gamma-\alpha-1} (1-zt)^{-\beta} dt.$$

Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$ be the upper half-plane. For real r_k and $h_k (k = 1, 2)$, let $\mathbf{r} = (r_1, r_2)$ and $\mathbf{h} = (h_1, h_2)$. For $\tau \in \mathbb{H}$ and $s_1, s_2 \in \mathbb{C}$ with $s = s_1 + s_2$, define

$$\mathcal{A}(\tau, s_1, s_2; \mathbf{r}, \mathbf{h}) = \sum_{m+r_1>0} \sum_{n-h_2>0} \frac{e^{2\pi i(mh_1 + ((m+r_1)\tau+r_2)(n-h_2))}}{(n-h_2)^{1-s}} \times U(s_2; s; 4\pi(m+r_1)(n-h_2)\text{Im}(\tau))$$

and

$$\bar{\mathcal{A}}(\tau, s_1, s_2; \mathbf{r}, \mathbf{h}) = \sum_{m+r_1>0} \sum_{n+h_2>0} \frac{e^{2\pi i(mh_1 - ((m+r_1)\bar{\tau}+r_2)(n+h_2))}}{(n+h_2)^{1-s}} \times U(s_1; s; 4\pi(m+r_1)(n+h_2)\text{Im}(\tau)).$$

Let

$$\mathcal{H}(\tau, s_1, s_2; \mathbf{r}, \mathbf{h}) = \mathcal{A}(\tau, s_1, s_2; \mathbf{r}, \mathbf{h}) + e^{\pi i s} \mathcal{A}(\tau, s_1, s_2; -\mathbf{r}, -\mathbf{h})$$

and

$$\bar{\mathcal{H}}(\tau, s_1, s_2; \mathbf{r}, \mathbf{h}) = \bar{\mathcal{A}}(\tau, s_1, s_2; \mathbf{r}, \mathbf{h}) + e^{\pi i s} \bar{\mathcal{A}}(\tau, s_1, s_2; -\mathbf{r}, -\mathbf{h}).$$

Let

$$\mathbf{H}(\tau, \bar{\tau}, s_1, s_2; \mathbf{r}, \mathbf{h}) = \frac{1}{\Gamma(s_1)} \mathcal{H}(\tau, s_1, s_2; \mathbf{r}, \mathbf{h}) + \frac{1}{\Gamma(s_2)} \bar{\mathcal{H}}(\tau, s_1, s_2; \mathbf{r}, \mathbf{h}).$$

For real x, y and $t \in \mathbb{C}$ with $\text{Re } t > 1$, let

$$\psi(x, y, t) = \sum_{n+y>0} \frac{e^{2\pi i n x}}{(n+y)^t}$$

and

$$\Psi(x, y, t) = \psi(x, y, t) + e^{\pi i t} \psi(-x, -y, t),$$

$$\Psi_{-1}(x, y, t) = \psi(x, y, t - 1) + e^{\pi it} \psi(-x, -y, t - 1).$$

The characteristic function of the integers is denoted by λ , i.e.,

$$\lambda(x) = \begin{cases} 1, & \text{if } x \text{ is an integer,} \\ 0, & \text{otherwise.} \end{cases}$$

For a real number x , $[x]$ denotes the greatest integer less than or equal to x and $\{x\} = x - [x]$. Let

$$V_\tau = \frac{a\tau + b}{c\tau + d}$$

denote a modular transformation with $c > 0$ for $\tau \in \mathbb{C}$. Let

$$R = (R_1, R_2) = (ar_1 + cr_2, br_1 + dr_2)$$

and

$$H = (H_1, H_2) = (dh_1 - bh_2, -ch_1 + ah_2).$$

The following theorem is the principal theorem which we shall use to obtain main results.

THEOREM 1.1. [4]. *Let $Q = \{\tau \in \mathbb{H} \mid \text{Re } \tau > -d/c\}$ and $\varrho = c\{R_2\} - d\{R_1\}$. Let $s_1, s_2 \in \mathbb{C}$ with $s = s_1 + s_2$ and assume that s is not an integer less than or equal to 1. Then, for $\tau \in Q$,*

$$\begin{aligned} z^{-s_1} \bar{z}^{-s_2} \mathbf{H}(V\tau, V\bar{\tau}, s_1, s_2; \mathbf{r}, \mathbf{h}) &= \mathbf{H}(\tau, \bar{\tau}, s_1, s_2; R, H) \\ &+ \lambda(R_1)(2\pi i)^{-s} e^{-\pi i(2R_1H_1+s_2)} \Psi(-H_2, -R_2, s) \\ &- \lambda(r_1)(2\pi i)^{-s} e^{-\pi i(2r_1h_1-s_1)} z^{-s_1} \bar{z}^{-s_2} \Psi(h_2, r_2, s) \\ &+ \lambda(H_2)(4\pi \text{Im}(\tau))^{1-s} \frac{\Gamma(s-1)}{\Gamma(s_1)\Gamma(s_2)} \Psi_{-1}(H_1, R_1, s) \\ &- \lambda(h_2)(4\pi \text{Im}(\tau))^{1-s} \frac{\Gamma(s-1)}{\Gamma(s_1)\Gamma(s_2)} z^{s_2-1} \bar{z}^{s_1-1} \Psi_{-1}(h_1, r_1, s) \\ &+ \frac{(2\pi i)^{-s} e^{-\pi is_2}}{\Gamma(s_1)\Gamma(s_2)} \mathbf{L}(\tau, \bar{\tau}, s_1, s_2; R, H), \end{aligned}$$

where $z = c\tau + d$ and

$$\begin{aligned} &\mathbf{L}(\tau, \bar{\tau}, s_1, s_2; R, H) \\ &= \sum_{j=1}^c e^{-2\pi i(H_1(j+[R_1]-c)+H_2([R_2]+1-[(jd+\varrho)/c]+d))} \\ &\quad \times \int_0^1 v^{s_1-1} (1-v)^{s_2-1} \int_C u^{s-1} \frac{e^{-(zv+\bar{z}(1-v))(j-\{R_1\})u/c}}{e^{-(zv+\bar{z}(1-v))u} - e^{2\pi i(cH_1+dH_2)}} \\ &\quad \times \frac{e^{\{(jd+\varrho)/c\}u}}{e^u - e^{-2\pi iH_2}} \, dudv, \end{aligned}$$

where C is a loop beginning at $+\infty$, proceeding in the upper half-plane, encircling the origin in the positive direction so that $u = 0$ is the only zero of

$$(e^{-(zv+\bar{z}(1-v))u} - e^{2\pi i(cH_1+dH_2)})(e^u - e^{-2\pi iH_2})$$

lying inside the loop and then returning to $+\infty$ in the lower half plane. Here, we choose the branch of u^s with $0 < \arg u < 2\pi$.

Let $s = s_1 + s_2$ be an integer. Suppose $H_1 = H_2 = 0$. The contour integral in Theorem 1.1 can be evaluated by using the residue theorem, i.e., we find that

$$\begin{aligned} & \int_C u^{s-1} \frac{e^{-(zv+\bar{z}(1-v))(j-\{R_1\})u/c} e^{\{\varrho+jd\}/c} u}{e^{-(zv+\bar{z}(1-v))u} - 1} \frac{e^{\{\varrho+jd\}/c} u}{e^u - 1} dudv \\ &= 2\pi i \sum_{k=0}^{-s+2} \frac{B_k((j-\{R_1\})/c) \bar{B}_{-s+2-k}(\{\varrho+jd\}/c)}{k!(-s+2-k)!} (-z)^{k-1}, \end{aligned}$$

where $B_n(x)$ is the n -th Bernoulli polynomial defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (|t| < 2\pi).$$

2. A few classes of infinite series identities

In this section, we obtain new infinite series identities using Theorem 1.1. The Eulerian number $E(n, j)$ is defined to be the number of permutations of numbers from 1 to n such that exactly j numbers are greater than the previous elements. Note that $E(n, j) = E(n, n - j - 1)$. For any integer n , the polylogarithm function $\text{Li}_n(z)$ is defined by

$$\text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n},$$

where $z \in \mathbb{C}$ and $|z| < 1$. For $n > 0$, we see that

$$\text{Li}_{-n}(z) = \frac{1}{(1-z)^{n+1}} \sum_{j=0}^{n-1} E(n, j) z^{n-j}.$$

Here, we set

$$V\tau = \frac{\tau - 1}{c\tau - c + 1},$$

where c is a positive integer. For any integer $k \geq 1$, let

$$\mu_k = \begin{cases} \frac{1}{2}, & k \text{ odd,} \\ 0, & k \text{ even.} \end{cases}$$

Put $\mathbf{r} = (r, 0)$, $0 < r < 1$, $\mathbf{h} = (0, 0)$ and $c\tau - c + 1 = \frac{\pi}{\alpha}i$. Let $s_1 = A \geq 1$, $s_2 = -B \leq 0$ and $s = s_1 + s_2 = 2N \geq 2$. Then, employing

$$\begin{aligned} & U(-B; 2N; 4\pi(m+r)n\text{Im}(V\tau)) \\ &= (-1)^B(A-1)! \sum_{k=0}^B \binom{B}{k} \frac{(-4\pi(m+r)n\text{Im}(V\tau))^k}{(2N-1+k)!}, \end{aligned}$$

we have

$$\begin{aligned} \mathcal{A}(V\tau, A, -B; \mathbf{r}, \mathbf{h}) &= \sum_{m+r>0} \sum_{n=1}^{\infty} \frac{e^{2\pi i(m+r)nV\tau}}{n^{1-2N}} \\ &\quad \times U(-B; 2N; 4\pi(m+r)n\text{Im}(V\tau)) \\ &= (-1)^B(A-1)! \sum_{k=0}^B \binom{B}{k} \frac{(-4\alpha/c)^k}{(2N+k-1)!} \\ &\quad \times \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{e^{-2(m+r)n(\alpha-i\pi)/c}}{(m+r)^{-k}n^{1-2N-k}} \\ &= (-1)^B(A-1)! \sum_{k=0}^B \binom{B}{k} \frac{(-4\alpha/c)^k}{(2N+k-1)!} \\ &\quad \times \sum_{m=0}^{\infty} \frac{\text{Li}_{(1-2N-k)}(e^{-2(m+r)(\alpha-i\pi)/c})}{(m+r)^{-k}}. \end{aligned}$$

Put $w = (m+r)(\alpha-i\pi)/c$. Applying $E(n, j) = E(n, n-j-1)$, we have

$$\begin{aligned} \text{Li}_{(1-2N-k)}(e^{-2w}) &= \frac{1}{(1-e^{-2w})^{2N+k}} \sum_{j=0}^{2N+k-2} \frac{E(2N+k-1, j)}{e^{2(2N+k-j-1)w}} \\ &= \frac{2^{1-2N-k}}{\sinh^{2N+k} w} \sum_{j=0}^{N+[k/2]-1} \cosh(2w(j+\mu_k)) \\ &\quad \times E(2N+k-1, N+[k/2]-j-1), \end{aligned}$$

where $'$ means that if k is odd, then the term with $j = 0$ is multiplied by $\frac{1}{2}$. Thus

$$\mathcal{A}(V\tau, A, -B; \mathbf{r}, \mathbf{h}) = \frac{(-1)^B(A-1)!}{2^{2N-1}} \sum_{k=0}^B \binom{B}{k} \frac{(-2\alpha/c)^k}{(2N+k-1)!}$$

$$\begin{aligned} & \times \sum_{j=0}^{N+[k/2]-1} E(2N+k-1, N+[k/2]-j-1) \\ & \times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+\mu_k)(\alpha-i\pi)/c)}{(m+r)^{-k} \sinh^{2N+k}((m+r)(\alpha-i\pi)/c)}. \end{aligned}$$

Similarly we have

$$\begin{aligned} \mathcal{A}(V\tau, A, -B; -\mathbf{r}, -\mathbf{h}) &= \frac{(-1)^B(A-1)!}{2^{2N-1}} \sum_{k=0}^B \binom{B}{k} \frac{(-2\alpha/c)^k}{(2N+k-1)!} \\ & \times \sum_{j=0}^{N+[k/2]-1} E(2N+k-1, N+[k/2]-j-1) \\ & \times \sum_{m=0}^{\infty} \frac{\cosh(2(m-r)(j+\mu_k)(\alpha-i\pi)/c)}{(m-r)^{-k} \sinh^{2N+k}((m-r)(\alpha-i\pi)/c)}. \end{aligned}$$

Hence we find

$$\begin{aligned} \mathbf{H}(V\tau, A, -B; \mathbf{r}, \mathbf{h}) &= \frac{1}{(A-1)!} (\mathcal{A}(V\tau, A, -B; \mathbf{r}, \mathbf{h}) + \mathcal{A}(V\tau, A, -B; -\mathbf{r}, -\mathbf{h})) \\ &= \frac{(-1)^B}{2^{2N-1}} \sum_{k=0}^B \binom{B}{k} \frac{(-2\alpha/c)^k}{(2N+k-1)!} \\ & \times \sum_{j=0}^{N+[k/2]-1} E(2N+k-1, N+[k/2]-j-1) \\ & \times \left(\sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+\mu_k)(\alpha-i\pi)/c)}{(m+r)^{-k} \sinh^{2N+k}((m+r)(\alpha-i\pi)/c)} \right. \\ & \left. + \sum_{m=0}^{\infty} \frac{\cosh(2(m-r)(j+\mu_k)(\alpha-i\pi)/c)}{(m-r)^{-k} \sinh^{2N+k}((m-r)(\alpha-i\pi)/c)} \right). \end{aligned}$$

By the same way, we obtain that

$$\begin{aligned} \mathcal{A}(\tau, A, -B; R, H) &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{e^{-2(m+r)n(\beta+i\pi)/c}}{n^{1-2N}} U(-B; 2N; 4\beta(m+r)n/c) \\ &= \frac{(-1)^B(A-1)!}{2^{2N-1}} \sum_{k=0}^B \binom{B}{k} \frac{(-2\beta/c)^k}{(2N+k-1)!} \\ & \times \sum_{j=0}^{N+[k/2]-1} E(2N+k-1, N+[k/2]-j-1) \end{aligned}$$

$$\begin{aligned} & \times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+\mu_k)(\beta+i\pi)/c)}{(m+r)^{-k} \sinh^{2N+k}((m+r)(\beta+i\pi)/c)}, \\ \mathcal{A}(\tau, A, -B; -R, -H) &= \frac{(-1)^B(A-1)!}{2^{2N-1}} \sum_{k=0}^B \binom{B}{k} \frac{(-2\beta/c)^k}{(2N+k-1)!} \\ & \times \sum_{j=0}^{N+[k/2]-1} E(2N+k-1, N+[k/2]-j-1) \\ & \times \sum_{m=0}^{\infty} \frac{\cosh(2(m-r)(j+\mu_k)(\beta+i\pi)/c)}{(m-r)^{-k} \sinh^{2N+k}((m-r)(\beta+i\pi)/c)} \end{aligned}$$

and

$$\begin{aligned} \mathbf{H}(\tau, A, -B; R, H) &= \frac{(-1)^B}{2^{2N-1}} \sum_{k=0}^B \binom{B}{k} \frac{(-2\beta/c)^k}{(2N+k-1)!} \\ & \times \sum_{j=0}^{N+[k/2]-1} E(2N+k-1, N+[k/2]-j-1) \\ & \times \left(\sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+\mu_k)(\beta+i\pi)/c)}{(m+r)^{-k} \sinh^{2N+k}((m+r)(\beta+i\pi)/c)} \right. \\ & \left. + \sum_{m=0}^{\infty} \frac{\cosh(2(m-r)(j+\mu_k)(\beta+i\pi)/c)}{(m-r)^{-k} \sinh^{2N+k}((m-r)(\beta+i\pi)/c)} \right). \end{aligned}$$

Using the residue theorem and the integral representation of the hypergeometric function, we have

$$\begin{aligned} & \frac{(2\pi i)^{-s} e^{-\pi i s_2}}{\Gamma(s_1)\Gamma(s_2)} \mathbf{L}(\tau, \bar{\tau}, s_1, s_2; R, H) \\ &= \frac{(-1)^B(2\pi i)^{1-2N}}{\Gamma(2N)} \sum_{j=1}^c \sum_{k=0}^{2-2N} \frac{B_k((j-\{R_1\})/c) \bar{B}_{2-2N-k}((\varrho+j(1-c))/c)}{k!(2-2N-k)!} \\ & \quad \times \left(-\frac{i\pi}{\alpha} \right)^{k-1} {}_2F_1(-B, 1-k, 2N; 2) \\ &= \begin{cases} \frac{c(1+(-1)^B)}{4\beta(B+1)}, & N=1, \\ 0, & N \geq 2. \end{cases} \end{aligned}$$

Finally, we obtain the following theorem.

THEOREM 2.1. *Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$ and let $0 < r < 1$. For any integers $B \geq 0$ and $N \geq 1$,*

$$\begin{aligned} & (-1)^B \alpha^N \sum_{k=0}^B \binom{B}{k} \frac{(-2\alpha/c)^k}{(2N+k-1)!} \sum_{j=0}^{N+[k/2]-1} E(2N+k-1, N+[k/2]-j-1) \\ & \quad \times \left(\sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+\mu_k)(\alpha-i\pi)/c)}{(m+r)^{-k} \sinh^{2N+k}((m+r)(\alpha-i\pi)/c)} \right. \\ & \quad \quad \left. + \sum_{m=0}^{\infty} \frac{\cosh(2(m-r)(j+\mu_k)(\alpha-i\pi)/c)}{(m-r)^{-k} \sinh^{2N+k}((m-r)(\alpha-i\pi)/c)} \right) \\ & = (-\beta)^N \sum_{k=0}^B \binom{B}{k} \frac{(-2\beta/c)^k}{(2N+k-1)!} \sum_{j=0}^{N+[k/2]-1} E(2N+k-1, N+[k/2]-j-1) \\ & \quad \times \left(\sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+\mu_k)(\beta+i\pi)/c)}{(m+r)^{-k} \sinh^{2N+k}((m+r)(\beta+i\pi)/c)} \right. \\ & \quad \quad \left. + \sum_{m=0}^{\infty} \frac{\cosh(2(m-r)(j+\mu_k)(\beta+i\pi)/c)}{(m-r)^{-k} \sinh^{2N+k}((m-r)(\beta+i\pi)/c)} \right) \\ & \quad - \nu_N(B, c), \end{aligned}$$

where

$$\nu_N(B, c) = \begin{cases} \frac{c(1+(-1)^B)}{2(B+1)}, & N = 1, \\ 0, & N \geq 2. \end{cases}$$

The case of $r = \frac{1}{2}$ in Theorem 2.1 is evaluated in [5]

COROLLARY 2.2. *Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. For any integer $N \geq 1$,*

$$\begin{aligned} & \alpha^N \sum_{j=0}^{N-1} E(2N-1, N-j-1) \\ & \quad \times \sum_{\substack{m \equiv 1, 2 \\ (\text{mod } 3)}} \frac{\cosh(2mj(\alpha-i\pi)/3)}{\sinh^{2N}(m(\alpha-i\pi)/3)} \\ & = (-\beta)^N \sum_{j=0}^{N-1} E(2N-1, N-j-1) \\ & \quad \times \sum_{\substack{m \equiv 1, 2 \\ (\text{mod } 3)}} \frac{\cosh(2mj(\beta+i\pi)/3)}{\sinh^{2N}(m(\beta+i\pi)/3)} - \nu_N(0, 1). \end{aligned}$$

Proof. Put $r = \frac{1}{3}$, $B = 0$ and $c = 1$ in Theorem 2.1. □

COROLLARY 2.3. We have

$$\begin{aligned} & \sum_{m=1}^{\infty} \operatorname{csch}^2(m\pi(1-i)/3) + \sum_{m=1}^{\infty} \operatorname{csch}^2(m\pi(1+i)/3) \\ &= 2 \sum_{m=1}^{\infty} \operatorname{csch}^2(m\pi) - \frac{2}{\pi}. \end{aligned}$$

Proof. Let $\alpha = \beta = \pi$ and $N = 1$ in Corollary 2.2. Then

$$\begin{aligned} & \sum_{m \equiv 1,2 \pmod{3}} \operatorname{csch}^2(m\pi(1-i)/3) \\ &= - \sum_{m \equiv 1,2 \pmod{3}} \operatorname{csch}^2(m\pi(1+i)/3) - \frac{2}{\pi}. \end{aligned}$$

Use $\sinh(m\pi(1 \pm i)) = (-1)^m \sinh(m\pi)$ to obtain the result. □

The identity in Corollary 2.3 can be written as a different form. By using

$$\operatorname{csch}(m\pi(1 \pm i)/3) = \frac{\sinh(m\pi/3) \cos(m\pi/3)}{\sinh^2(m\pi/3) + \sin^2(m\pi/3)} \mp i \frac{\cosh(m\pi/3) \sin(m\pi/3)}{\sinh^2(m\pi/3) + \sin^2(m\pi/3)},$$

we see that

$$\begin{aligned} & \operatorname{csch}^2(m\pi(1-i)/3) + \operatorname{csch}^2(m\pi(1+i)/3) \\ &= 2 \frac{\sinh^2(m\pi/3) \cos^2(m\pi/3) - \cosh^2(m\pi/3) \sin^2(m\pi/3)}{(\sinh^2(m\pi/3) + \sin^2(m\pi/3))^2} \\ &= 2 \frac{\sinh^2(m\pi/3) - \cosh(2m\pi/3) \sin^2(m\pi/3)}{(\sinh^2(m\pi/3) + \sin^2(m\pi/3))^2}. \end{aligned}$$

Applying $\sin^2(\frac{m\pi}{3}) = \frac{3}{4}$ for $m \equiv 1, 2 \pmod{3}$, we have

$$\begin{aligned} & \sum_{m \equiv 1 \pmod{3}} \frac{4 \sinh^2(m\pi/3) - 3 \cosh(2m\pi/3)}{(4 \sinh^2(m\pi/3) + 3)^2} \\ &= - \sum_{m \equiv 2 \pmod{3}} \frac{4 \sinh^2(m\pi/3) - 3 \cosh(2m\pi/3)}{(4 \sinh^2(m\pi/3) + 3)^2} - \frac{1}{4\pi}. \end{aligned}$$

Now we compute another infinite series identity by changing the values of s_1, s_2 . Put $\mathbf{r} = (r, 0)$, $0 < r < 1$ and $\mathbf{h} = (0, 0)$. Let $s_1 = A + 1 \geq 1$, $s_2 = B + 1 \geq 1$ and $s = s_1 + s_2 = 2N + 2 \geq 4$. In this

case, for any integer $k \geq 1$, we put

$$\hat{k} = \left[\frac{k-1}{2} \right], \quad v_k = \begin{cases} 0, & k \text{ odd,} \\ \frac{1}{2}, & k \text{ even.} \end{cases}$$

Let $z = c\tau - c + 1$ and put $z = \frac{\pi}{\alpha}i$. Then, using

$$e^{2\pi i((m+r)\tau-r)n} = e^{-2n(m+r)(\beta+i\pi)/c}$$

and

$$\begin{aligned} & U(B+1; 2N+2; 4\pi(m+r)n\text{Im}(\tau)) \\ &= \frac{1}{B!} \sum_{k=0}^A \binom{A}{k} \frac{(2N-k)!}{(4\pi(m+r)n\beta/c)^{2N-k+1}}, \end{aligned}$$

we have

$$\begin{aligned} \mathcal{A}(\tau, s_1, s_2; R, H) &= \frac{1}{B!} \sum_{k=0}^A \binom{A}{k} \frac{(2N-k)!}{(4\beta/c)^{2N-k+1}} \sum_{m=0}^{\infty} \frac{\text{Li}_{-k}(e^{-2(m+r)(\beta+i\pi)/c})}{(m+r)^{2N-k+1}} \\ &= \frac{1}{B!} \sum_{k=1}^A \binom{A}{k} \frac{2^{-k}(2N-k)!}{(4\beta/c)^{2N-k+1}} \sum_{j=0}^{\hat{k}'} E(k, \hat{k}-j) \\ &\quad \times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+v_k)(\beta+i\pi)/c)}{(m+r)^{2N-k+1} \sinh^{k+1}((m+r)(\beta+i\pi)/c)} \\ &\quad + \frac{(2N)!}{B!(4\beta/c)^{2N+1}} \sum_{m=0}^{\infty} \frac{(m+r)^{-2N-1}}{e^{2(m+r)(\beta+i\pi)/c} - 1}, \end{aligned}$$

where ' means that if k is odd, then the term with $j = 0$ is multiplied by $\frac{1}{2}$. By the same way, other functions can be evaluated, namely, we have

$$\begin{aligned} \mathcal{A}(\tau, s_1, s_2; -R, -H) &= \frac{1}{B!} \sum_{k=1}^A \binom{A}{k} \frac{2^{-k}(2N-k)!}{(4\beta/c)^{2N-k+1}} \sum_{j=0}^{\hat{k}'} E(k, \hat{k}-j) \\ &\quad \times \sum_{m=1}^{\infty} \frac{\cosh(2(m-r)(j+v_k)(\beta+i\pi)/c)}{(m-r)^{2N-k+1} \sinh^{k+1}((m-r)(\beta+i\pi)/c)} \\ &\quad + \frac{(2N)!}{B!(4\beta/c)^{2N+1}} \sum_{m=1}^{\infty} \frac{(m-r)^{-2N-1}}{e^{2(m-r)(\beta+i\pi)/c} - 1}, \end{aligned}$$

$$\bar{\mathcal{A}}(\tau, s_1, s_2; R, H) = \frac{1}{A!} \sum_{k=1}^B \binom{B}{k} \frac{2^{-k}(2N-k)!}{(4\beta/c)^{2N-k+1}} \sum_{j=0}^{\hat{k}'} E(k, \hat{k}-j)$$

$$\begin{aligned} & \times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+v_k)(\beta-i\pi)/c)}{(m+r)^{2N-k+1} \sinh^{k+1}((m+r)(\beta-i\pi)/c)} \\ & + \frac{(2N)!}{A!(4\beta/c)^{2N+1}} \sum_{m=0}^{\infty} \frac{(m+r)^{-2N-1}}{e^{2(m+r)(\beta-i\pi)/c} - 1}, \end{aligned}$$

$$\begin{aligned} \bar{\mathcal{A}}(\tau, s_1, s_2; -R, -H) &= \frac{1}{A!} \sum_{k=1}^B \binom{B}{k} \frac{2^{-k}(2N-k)!}{(4\beta/c)^{2N-k+1}} \sum_{j=0}^{\hat{k}'} E(k, \hat{k}-j) \\ & \times \sum_{m=1}^{\infty} \frac{\cosh(2(m-r)(j+v_k)(\beta-i\pi)/c)}{(m-r)^{2N-k+1} \sinh^{k+1}((m-r)(\beta-i\pi)/c)} \\ & + \frac{(2N)!}{A!(4\beta/c)^{2N+1}} \sum_{m=1}^{\infty} \frac{(m-r)^{-2N-1}}{e^{2(m-r)(\beta-i\pi)/c} - 1}, \end{aligned}$$

$$\begin{aligned} \mathcal{A}(V\tau, s_1, s_2; \mathbf{r}, \mathbf{h}) &= \frac{1}{B!} \sum_{k=1}^A \binom{A}{k} \frac{2^{-k}(2N-k)!}{(4\alpha/c)^{2N-k+1}} \sum_{j=0}^{\hat{k}'} E(k, \hat{k}-j) \\ & \times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+v_k)(\alpha-i\pi)/c)}{(m+r)^{2N-k+1} \sinh^{k+1}((m+r)(\alpha-i\pi)/c)} \\ & + \frac{(2N)!}{B!(4\alpha/c)^{2N+1}} \sum_{m=0}^{\infty} \frac{(m+r)^{-2N-1}}{e^{2(m+r)(\alpha-i\pi)/c} - 1}, \end{aligned}$$

$$\begin{aligned} \mathcal{A}(V\tau, s_1, s_2; -\mathbf{r}, -\mathbf{h}) &= \frac{1}{B!} \sum_{k=1}^A \binom{A}{k} \frac{2^{-k}(2N-k)!}{(4\alpha/c)^{2N-k+1}} \sum_{j=0}^{\hat{k}'} E(k, \hat{k}-j) \\ & \times \sum_{m=1}^{\infty} \frac{\cosh(2(m-r)(j+v_k)(\alpha-i\pi)/c)}{(m-r)^{2N-k+1} \sinh^{k+1}((m-r)(\alpha-i\pi)/c)} \\ & + \frac{(2N)!}{B!(4\alpha/c)^{2N+1}} \sum_{m=1}^{\infty} \frac{(m-r)^{-2N-1}}{e^{2(m-r)(\alpha-i\pi)/c} - 1}, \end{aligned}$$

$$\begin{aligned} \bar{\mathcal{A}}(V\tau, s_1, s_2; \mathbf{r}, \mathbf{h}) &= \frac{1}{A!} \sum_{k=1}^B \binom{B}{k} \frac{2^{-k}(2N-k)!}{(4\alpha/c)^{2N-k+1}} \sum_{j=0}^{\hat{k}'} E(k, \hat{k}-j) \\ & \times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+v_k)(\alpha+i\pi)/c)}{(m+r)^{2N-k+1} \sinh^{k+1}((m+r)(\alpha+i\pi)/c)} \end{aligned}$$

$$\begin{aligned}
 & + \frac{(2N)!}{A!(4\alpha/c)^{2N+1}} \sum_{m=0}^{\infty} \frac{(m+r)^{-2N-1}}{e^{2(m+r)(\alpha+i\pi)/c} - 1}, \\
 \bar{A}(V\tau, s_1, s_2; -\mathbf{r}, -\mathbf{h}) & = \frac{1}{A!} \sum_{k=1}^B \binom{B}{k} \frac{2^{-k}(2N-k)!}{(4\alpha/c)^{2N-k+1}} \sum_{j=0}^{\hat{k}} E(k, \hat{k} - j) \\
 & \times \sum_{m=1}^{\infty} \frac{\cosh(2(m-r)(j+v_k)(\alpha+i\pi)/c)}{(m-r)^{2N-k+1} \sinh^{k+1}((m-r)(\alpha+i\pi)/c)} \\
 & + \frac{(2N)!}{A!(4\alpha/c)^{2N+1}} \sum_{m=1}^{\infty} \frac{(m-r)^{-2N-1}}{e^{2(m-r)(\alpha+i\pi)/c} - 1}.
 \end{aligned}$$

It is easy to see that

$$\begin{aligned}
 \Psi_{-1}(H_1, R_1, s) & = \psi(0, r, 2N+1) + \psi(0, -r, 2N+1) \\
 & = \zeta(2N+1, r) + \zeta(2N+1, 1-r),
 \end{aligned}$$

$$\begin{aligned}
 \Psi_{-1}(h_1, r_1, s) & = \psi(0, r, 2N+1) + \psi(0, -r, 2N+1) \\
 & = \zeta(2N+1, r) + \zeta(2N+1, 1-r),
 \end{aligned}$$

$$z^{-s_1} \bar{z}^{-s_2} = (-1)^{B+1} \alpha^{N+1} (-\beta)^{-N-1},$$

$$z^{s_2-1} \bar{z}^{s_1-1} = (-1)^B \alpha^{-N} (-\beta)^N.$$

Note that $s = 2N + 2 \geq 4$. Then, by using the residue theorem, we have

$$\frac{1}{\Gamma(s_1)\Gamma(s_2)} \mathbf{L}(\tau, \bar{\tau}, s_1, s_2; R, H) = 0.$$

We now obtain the following theorem.

THEOREM 2.4. *Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$ and let $0 < r < 1$. For any integers $A \geq 0, B \geq 0$ and $N \geq 1$ with $A + B = 2N$,*

$$\begin{aligned}
 & (-1)^B \alpha^{-N} \sum_{k=1}^A \binom{A}{k} \frac{(2N-k)!}{(2\alpha/c)^{-k}} \sum_{j=0}^{\hat{k}} E(k, \hat{k} - j) \\
 & \times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+v_k)(\alpha-i\pi)/c)}{(m+r)^{2N-k+1} \sinh^{k+1}((m+r)(\alpha-i\pi)/c)} \\
 & + (-1)^B (2N)! \alpha^{-N} \sum_{m=0}^{\infty} \frac{(m+r)^{-2N-1}}{e^{2(m+r)(\alpha-i\pi)/c} - 1} \\
 & + (-1)^B \alpha^{-N} \sum_{k=1}^A \binom{A}{k} \frac{(2N-k)!}{(2\alpha/c)^{-k}} \sum_{j=0}^{\hat{k}} E(k, \hat{k} - j)
 \end{aligned}$$

$$\begin{aligned}
 & \times \sum_{m=1}^{\infty} \frac{\cosh(2(m-r)(j+v_k)(\alpha-i\pi)/c)}{(m-r)^{2N-k+1} \sinh^{k+1}((m-r)(\alpha-i\pi)/c)} \\
 & + (-1)^B (2N)! \alpha^{-N} \sum_{m=1}^{\infty} \frac{(m-r)^{-2N-1}}{e^{2(m-r)(\alpha-i\pi)/c} - 1} \\
 & + (-1)^B \alpha^{-N} \sum_{k=1}^B \binom{B}{k} \frac{(2N-k)!}{(2\alpha/c)^{-k}} \sum_{j=0}^{\hat{k}} E(k, \hat{k}-j) \\
 & \times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+v_k)(\alpha+i\pi)/c)}{(m+r)^{2N-k+1} \sinh^{k+1}((m+r)(\alpha+i\pi)/c)} \\
 & + (-1)^B (2N)! \alpha^{-N} \sum_{m=0}^{\infty} \frac{(m+r)^{-2N-1}}{e^{2(m+r)(\alpha+i\pi)/c} - 1} \\
 & + (-1)^B \alpha^{-N} \sum_{k=1}^B \binom{B}{k} \frac{(2N-k)!}{(2\alpha/c)^{-k}} \sum_{j=0}^{\hat{k}} E(k, \hat{k}-j) \\
 & \times \sum_{m=1}^{\infty} \frac{\cosh(2(m-r)(j+v_k)(\alpha+i\pi)/c)}{(m-r)^{2N-k+1} \sinh^{k+1}((m-r)(\alpha+i\pi)/c)} \\
 & + (-1)^B (2N)! \alpha^{-N} \sum_{m=1}^{\infty} \frac{(m-r)^{-2N-1}}{e^{2(m-r)(\alpha+i\pi)/c} - 1} \\
 & + (-1)^B (2N)! \alpha^{-N} (\zeta(2N+1, r) + \zeta(2N+1, 1-r)) \\
 & = (-1)^N \beta^{-N} \sum_{k=1}^A \binom{A}{k} \frac{(2N-k)!}{(2\beta/c)^{-k}} \sum_{j=0}^{\hat{k}} E(k, \hat{k}-j) \\
 & \times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+v_k)(\beta+i\pi)/c)}{(m+r)^{2N-k+1} \sinh^{k+1}((m+r)(\beta+i\pi)/c)} \\
 & + (-1)^N (2N)! \beta^{-N} \sum_{m=0}^{\infty} \frac{(m+r)^{-2N-1}}{e^{2(m+r)(\beta+i\pi)/c} - 1} \\
 & + (-1)^N \beta^{-N} \sum_{k=1}^A \binom{A}{k} \frac{(2N-k)!}{(2\beta/c)^{-k}} \sum_{j=0}^{\hat{k}} E(k, \hat{k}-j) \\
 & \times \sum_{m=1}^{\infty} \frac{\cosh(2(m-r)(j+v_k)(\beta+i\pi)/c)}{(m-r)^{2N-k+1} \sinh^{k+1}((m-r)(\beta+i\pi)/c)} \\
 & + (-1)^N (2N)! \beta^{-N} \sum_{m=1}^{\infty} \frac{(m-r)^{-2N-1}}{e^{2(m-r)(\beta+i\pi)/c} - 1}
 \end{aligned}$$

$$\begin{aligned}
& +(-1)^N \beta^{-N} \sum_{k=1}^B \binom{B}{k} \frac{(2N-k)!}{(2\beta/c)^{-k}} \sum_{j=0}^{\hat{k}}' E(k, \hat{k}-j) \\
& \quad \times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+v_k)(\beta-i\pi)/c)}{(m+r)^{2N-k+1} \sinh^{k+1}((m+r)(\beta-i\pi)/c)} \\
& \quad + (-1)^N (2N)! \beta^{-N} \sum_{m=0}^{\infty} \frac{(m+r)^{-2N-1}}{e^{2(m+r)(\beta-i\pi)/c} - 1} \\
& +(-1)^N \beta^{-N} \sum_{k=1}^B \binom{B}{k} \frac{(2N-k)!}{(2\beta/c)^{-k}} \sum_{j=0}^{\hat{k}}' E(k, \hat{k}-j) \\
& \quad \times \sum_{m=1}^{\infty} \frac{\cosh(2(m-r)(j+v_k)(\beta-i\pi)/c)}{(m-r)^{2N-k+1} \sinh^{k+1}((m-r)(\beta-i\pi)/c)} \\
& \quad + (-1)^N (2N)! \beta^{-N} \sum_{m=1}^{\infty} \frac{(m-r)^{-2N-1}}{e^{2(m-r)(\beta-i\pi)/c} - 1} \\
& +(-1)^N (2N)! \beta^{-N} (\zeta(2N+1, r) + \zeta(2N+1, 1-r)).
\end{aligned}$$

Note that the series in Theorem 2.4 contains sums of complex conjugates, which shows they are real. A specific case ($r = \frac{1}{2}$) of Theorem 2.4 is established in [6]

COROLLARY 2.5. *Let $0 < r < 1$ and let $A \not\equiv B \pmod{4}$. Then*

$$\begin{aligned}
& \sum_{k=1}^A \binom{A}{k} \frac{(2N-k)!}{(2\pi/c)^{-k}} \sum_{j=0}^{\hat{k}}' E(k, \hat{k}-j) \\
& \quad \times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+v_k)(\pi-i\pi)/c)}{(m+r)^{2N-k+1} \sinh^{k+1}((m+r)(\pi-i\pi)/c)} \\
& + \sum_{k=1}^A \binom{A}{k} \frac{(2N-k)!}{(2\pi/c)^{-k}} \sum_{j=0}^{\hat{k}}' E(k, \hat{k}-j) \\
& \quad \times \sum_{m=1}^{\infty} \frac{\cosh(2(m-r)(j+v_k)(\pi-i\pi)/c)}{(m-r)^{2N-k+1} \sinh^{k+1}((m-r)(\pi-i\pi)/c)} \\
& + \sum_{k=1}^B \binom{B}{k} \frac{(2N-k)!}{(2\pi/c)^{-k}} \sum_{j=0}^{\hat{k}}' E(k, \hat{k}-j) \\
& \quad \times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+v_k)(\pi+i\pi)/c)}{(m+r)^{2N-k+1} \sinh^{k+1}((m+r)(\pi+i\pi)/c)}
\end{aligned}$$

$$\begin{aligned}
 & + \sum_{k=1}^B \binom{B}{k} \frac{(2N-k)!}{(2\pi/c)^{-k}} \sum_{j=0}^{\hat{k}} E(k, \hat{k}-j) \\
 & \quad \times \sum_{m=1}^{\infty} \frac{\cosh(2(m-r)(j+v_k)(\pi+i\pi)/c)}{(m-r)^{2N-k+1} \sinh^{k+1}((m-r)(\pi+i\pi)/c)} \\
 & + \sum_{m=0}^{\infty} \frac{(2N)!}{(m+r)^{2N+1}} \frac{\cos(2\pi(m+r)/c) - e^{-2\pi(m+r)/c}}{\cosh(2\pi(m+r)/c) - \cos(2\pi(m+r)/c)} \\
 & + \sum_{m=1}^{\infty} \frac{(2N)!}{(m-r)^{2N+1}} \frac{\cos(2\pi(m-r)/c) - e^{-2\pi(m-r)/c}}{\cosh(2\pi(m-r)/c) - \cos(2\pi(m-r)/c)} \\
 & = \frac{(2N)!}{2} (\zeta(2N+1, r) + \zeta(2N+1, 1-r)).
 \end{aligned}$$

Proof. Put $\alpha = \beta = \pi$ in Theorem 2.4 and assume that B and N have the different parity. Apply

$$\frac{1}{e^{x+iy} - 1} + \frac{1}{e^{x-iy} - 1} = \frac{\cos y - e^{-x}}{\cosh x - \cos y}$$

to obtain the desired result. □

COROLLARY 2.6. *Let $0 < r < 1$. For any positive integer N ,*

$$\begin{aligned}
 & \sum_{k=1}^{4N-2} \frac{(2\pi)^k}{k!} \sum_{j=0}^{\hat{k}} E(k, \hat{k}-j) \\
 & \quad \times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+v_k)(\pi-i\pi))}{(m+r)^{4N-k-1} \sinh^{k+1}((m+r)(\pi-i\pi))} \\
 & + \sum_{k=1}^{4N-2} \frac{(2\pi)^k}{k!} \sum_{j=0}^{\hat{k}} E(k, \hat{k}-j) \\
 & \quad \times \sum_{m=1}^{\infty} \frac{\cosh(2(m-r)(j+v_k)(\pi-i\pi))}{(m-r)^{4N-k-1} \sinh^{k+1}((m-r)(\pi-i\pi))} \\
 & + \sum_{m=0}^{\infty} \frac{1}{(m+r)^{4N-1}} \frac{\cos(2\pi r) - e^{-2\pi(m+r)}}{\cosh(2\pi(m+r)) - \cos(2\pi r)} \\
 & + \sum_{m=1}^{\infty} \frac{1}{(m-r)^{4N-1}} \frac{\cos(2\pi r) - e^{-2\pi(m-r)}}{\cosh(2\pi(m-r)) - \cos(2\pi r)} \\
 & = \frac{1}{2} (\zeta(4N-1, r) + \zeta(4N-1, 1-r)).
 \end{aligned}$$

Proof. Put $c = 1$, $B = 0$ and $\alpha = \beta = \pi$ in Theorem 2.4. Replace N by $2N - 1$. □

The following two corollaries show elegant symmetric series relations with respect to α and β .

COROLLARY 2.7. *Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$ and let $0 < r < 1$. For any integer $N \geq 1$,*

$$\begin{aligned}
& \alpha^{-N} \sum_{k=1}^N \binom{N}{k} \frac{(2N-k)!}{(2\alpha/c)^{-k}} \sum_{j=0}^{\hat{k}} E(k, \hat{k}-j) \\
& \quad \times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+v_k)(\alpha-i\pi)/c)}{(m+r)^{2N-k+1} \sinh^{k+1}((m+r)(\alpha-i\pi)/c)} \\
& \quad + (2N)! \alpha^{-N} \sum_{m=0}^{\infty} \frac{(m+r)^{-2N-1}}{e^{2(m+r)(\alpha-i\pi)/c} - 1} \\
& + \alpha^{-N} \sum_{k=1}^N \binom{N}{k} \frac{(2N-k)!}{(2\alpha/c)^{-k}} \sum_{j=0}^{\hat{k}} E(k, \hat{k}-j) \\
& \quad \times \sum_{m=1}^{\infty} \frac{\cosh(2(m-r)(j+v_k)(\alpha-i\pi)/c)}{(m-r)^{2N-k+1} \sinh^{k+1}((m-r)(\alpha-i\pi)/c)} \\
& \quad + (2N)! \alpha^{-N} \sum_{m=1}^{\infty} \frac{(m-r)^{-2N-1}}{e^{2(m-r)(\alpha-i\pi)/c} - 1} \\
& + \alpha^{-N} \sum_{k=1}^N \binom{N}{k} \frac{(2N-k)!}{(2\alpha/c)^{-k}} \sum_{j=0}^{\hat{k}} E(k, \hat{k}-j) \\
& \quad \times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+v_k)(\alpha+i\pi)/c)}{(m+r)^{2N-k+1} \sinh^{k+1}((m+r)(\alpha+i\pi)/c)} \\
& \quad + (2N)! \alpha^{-N} \sum_{m=0}^{\infty} \frac{(m+r)^{-2N-1}}{e^{2(m+r)(\alpha+i\pi)/c} - 1} \\
& + \alpha^{-N} \sum_{k=1}^N \binom{N}{k} \frac{(2N-k)!}{(2\alpha/c)^{-k}} \sum_{j=0}^{\hat{k}} E(k, \hat{k}-j) \\
& \quad \times \sum_{m=1}^{\infty} \frac{\cosh(2(m-r)(j+v_k)(\alpha+i\pi)/c)}{(m-r)^{2N-k+1} \sinh^{k+1}((m-r)(\alpha+i\pi)/c)} \\
& \quad + (2N)! \alpha^{-N} \sum_{m=1}^{\infty} \frac{(m-r)^{-2N-1}}{e^{2(m-r)(\alpha+i\pi)/c} - 1} \\
& + (2N)! \alpha^{-N} (\zeta(2N+1, r) + \zeta(2N+1, 1-r))
\end{aligned}$$

$$\begin{aligned}
 &= \beta^{-N} \sum_{k=1}^N \binom{N}{k} \frac{(2N-k)!}{(2\beta/c)^{-k}} \sum_{j=0}^{\hat{k}'} E(k, \hat{k} - j) \\
 &\quad \times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+v_k)(\beta+i\pi)/c)}{(m+r)^{2N-k+1} \sinh^{k+1}((m+r)(\beta+i\pi)/c)} \\
 &\quad + (2N)! \beta^{-N} \sum_{m=0}^{\infty} \frac{(m+r)^{-2N-1}}{e^{2(m+r)(\beta+i\pi)/c} - 1} \\
 &+ \beta^{-N} \sum_{k=1}^N \binom{N}{k} \frac{(2N-k)!}{(2\beta/c)^{-k}} \sum_{j=0}^{\hat{k}'} E(k, \hat{k} - j) \\
 &\quad \times \sum_{m=1}^{\infty} \frac{\cosh(2(m-r)(j+v_k)(\beta+i\pi)/c)}{(m-r)^{2N-k+1} \sinh^{k+1}((m-r)(\beta+i\pi)/c)} \\
 &\quad + (2N)! \beta^{-N} \sum_{m=1}^{\infty} \frac{(m-r)^{-2N-1}}{e^{2(m-r)(\beta+i\pi)/c} - 1} \\
 &+ \beta^{-N} \sum_{k=1}^N \binom{N}{k} \frac{(2N-k)!}{(2\beta/c)^{-k}} \sum_{j=0}^{\hat{k}'} E(k, \hat{k} - j) \\
 &\quad \times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+v_k)(\beta-i\pi)/c)}{(m+r)^{2N-k+1} \sinh^{k+1}((m+r)(\beta-i\pi)/c)} \\
 &\quad + (2N)! \beta^{-N} \sum_{m=0}^{\infty} \frac{(m+r)^{-2N-1}}{e^{2(m+r)(\beta-i\pi)/c} - 1} \\
 &+ \beta^{-N} \sum_{k=1}^N \binom{N}{k} \frac{(2N-k)!}{(2\beta/c)^{-k}} \sum_{j=0}^{\hat{k}'} E(k, \hat{k} - j) \\
 &\quad \times \sum_{m=1}^{\infty} \frac{\cosh(2(m-r)(j+v_k)(\beta-i\pi)/c)}{(m-r)^{2N-k+1} \sinh^{k+1}((m-r)(\beta-i\pi)/c)} \\
 &\quad + (2N)! \beta^{-N} \sum_{m=1}^{\infty} \frac{(m-r)^{-2N-1}}{e^{2(m-r)(\beta-i\pi)/c} - 1} \\
 &+ (2N)! \beta^{-N} (\zeta(2N+1, r) + \zeta(2N+1, 1-r)).
 \end{aligned}$$

Proof. Let $A = B$ in Theorem 2.4. □

COROLLARY 2.8. Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$ and let $0 < r < 1$. Then

$$\sum_{m=0}^{\infty} \frac{2 \cos(2\pi r) \cosh(2(m+r)\alpha)}{(m+r)^2 (\cosh(2(m+r)\alpha) - \cos(2\pi r))^2}$$

$$\begin{aligned}
& + \sum_{m=1}^{\infty} \frac{2 \cos(2\pi r) \cosh(2(m-r)\alpha)}{(m-r)^2 (\cosh(2(m-r)\alpha) - \cos(2\pi r))^2} \\
& + \alpha^{-1} \sum_{m=0}^{\infty} \frac{\cos(2\pi r) - e^{-2(m+r)\alpha}}{(m+r)^3 (\cosh(2(m+r)\alpha) - \cos(2\pi r))} \\
& \alpha^{-1} \sum_{m=1}^{\infty} \frac{\cos(2\pi r) - e^{-2(m-r)\alpha}}{(m-r)^3 (\cosh(2(m-r)\alpha) - \cos(2\pi r))} \\
& + \alpha^{-1} (\zeta(3, r) + \zeta(3, 1-r)) \\
& = \sum_{m=0}^{\infty} \frac{2 \cos(2\pi r) \cosh(2(m+r)\beta)}{(m+r)^2 (\cosh(2(m+r)\beta) - \cos(2\pi r))^2} \\
& + \sum_{m=1}^{\infty} \frac{2 \cos(2\pi r) \cosh(2(m-r)\beta)}{(m-r)^2 (\cosh(2(m-r)\beta) - \cos(2\pi r))^2} \\
& + \beta^{-1} \sum_{m=0}^{\infty} \frac{\cos(2\pi r) - e^{-2(m+r)\beta}}{(m+r)^3 (\cosh(2(m+r)\beta) - \cos(2\pi r))} \\
& \beta^{-1} \sum_{m=1}^{\infty} \frac{\cos(2\pi r) - e^{-2(m-r)\beta}}{(m-r)^3 (\cosh(2(m-r)\beta) - \cos(2\pi r))} \\
& + \beta^{-1} (\zeta(3, r) + \zeta(3, 1-r)).
\end{aligned}$$

Proof. Put $A = B = 1$ and $c = 1$ in Corollary 2.7. Using

$$\sinh^2(x \pm iy) = \frac{1}{2}(\cosh(2x \pm i2y) - 1)$$

and

$$\cosh(x \pm iy) = \cosh x \cos y \pm i \sinh x \sin y,$$

the proof is done. \square

References

- [1] G. E. Andrews, R. Askey and R. Roy, *Special functions*, Cambridge University Press, 1999.
- [2] B. C. Berndt, *Modular transformations and generalizations of several formulae of Ramanujan*, The Rocky mountain J. Math., **7** (1977), no. 1, 147-189.
- [3] B. C. Berndt, *Analytic Eisenstein series, theta-functions, and series relations in the spirit of Ramanujan*, J. Für die reine und angewandte Math., **304** (1978), 332-365.
- [4] S. Lim, *Modular transformation formulae for generalized non-holomorphic Eisenstein series*, Honam Math. J., **35** (2013), no. 3, 507-513.
- [5] S. Lim, *Series relations coming from certain functions related to generalized non-holomorphic Eisenstein series*, J. Chungcheong Math. Soc., **34** (2021), no. 2, 139-155.
- [6] S. Lim, *Infinite series identities which stem from generalized non-holomorphic Eisenstein series*, J. Chungcheong Math. Soc., **32** (2019), no. 4, 421-440.

- [7] S. Ramanujan, *Notebooks of Srinivasa Ramanujan* (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.

Sung Geun Lim
Department of Mathematics Education
Mokwon University
Daejun 35356, Republic of Korea
E-mail: `sglimj@mokwon.ac.kr`